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Squeezing flow of fibre-reinforced viscous fluids

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Abstract. A relatively simple continuum model is described for the viscous flow of highly anisotropic materials such as fibre-reinforced resins. The theory is applied to the flow of such a fluid when squeezed between two rigid platens.

1. Introduction

In this note we present a flow analysis based on the concept of a *highly anisotropic viscous fluid.* For such a material the extensional viscosity (or modulus) in a given preferred direction is much greater than its shear and transverse extensional viscosities (or moduli). This behaviour is a natural characteristic of materials such as fibre-reinforced resins at elevated temperatures; whilst the composite is relatively free to distort and flow in directions that are locally transverse to the fibres and to shear along the fibres, extensional flow in the fibre direction is severely constrained by the fibres themselves.

The theory for highly anisotropic *solids* is now well-developed for linear and non-linear elastic and plastic behaviour. However, relatively little has been done in viscoelasticity and none for fluids.

A particularly useful model for solids has been the ideal fibre-reinforced material (e.g., [1], [2]). This is not only incompressible but also *inextensible* in the fibre-direction; furthermore, the constraint is assumed to be continuously distributed through the composite and to be convected with the deformation. This simple model has proved to be extremely useful in obtaining good approximate solutions to a large number of boundary-value problems for both small and large deformations, especially for those under plane strain or axisymmetric conditions. It has also highlighted the significant and occasionally surprising differences in solution brought about by the high anisotropy as compared with conventional isotropic or weakly anisotropic behaviour. These "ideal" solutions have been established as "outer" asymptotic solutions of the more accurate, but more complex, equations of linear and non-linear transversely isotropic elasticity with small extensibility in the preferred (fibre) direction. Accordingly the predictions of the ideal theory can be meaningfully interpreted and assessed for "real" material behaviour.

It is therefore anticipated that similar simplified, but useful, analysis can be obtained for highly anisotropic viscous fluids, and in this note we illustrate such a solution for the important practical problem of transverse flow induced by squeezing the fluid between two rigid platens ([3–5]). For simplicity, and since many resins behave as Newtonian fluids, we restrict attention to linear viscous fluids.

2. Fibre-reinforced linear viscous fluids

The constitutive equation for a conventional viscous fluid gives the state of stress $\boldsymbol{\sigma}$ at a point $\mathbf{x}$ and time t in terms of the rate-of-strain $\mathbf{d}$ at that same point and time. For an incompressible Newtonian fluid, the relation is linear and takes the familiar form

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{d}, \quad \operatorname{tr}\mathbf{d} = 0, \tag{1}$$

or, in terms of Cartesian components,

$$\sigma_{ij} = -p\delta_{ij} + 2\eta d_{ij}, \quad d_{ii} = 0, \quad (i, j = 1, 2, 3), \tag{2}$$

where the usual convention of summation over repeated indices is adopted. Here η denotes the (shear) viscosity of the fluid, p is the hydrostatic pressure and $\mathbf{d}$ is related to the velocity $\mathbf{v}$ through

$$\mathbf{d} = \tfrac{1}{2}\{(\nabla\mathbf{v}) + (\nabla\mathbf{v})^{\mathrm{T}}\}, \quad d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{3}$$

The pressure is an arbitrary function of $\mathbf{x}$ and t and is the reaction to the internal constraint of incompressibility.

The presence of relatively strong fibres radically alters the response of the fluid, introducing a preferred "strong" direction $\mathbf{a}$ at every point. By assuming that the stress $\boldsymbol{\sigma}$ will now depend not only on $\mathbf{d}$ but also on $\mathbf{a}$, it is straightforward to show that the most general *linear* stress-strain-rate relation must be

$$\sigma_{ij} = -p\delta_{ij} + 2\eta_T d_{ij} + 2(\eta_L - \eta_T)(a_i a_k d_{kj} + a_j a_k d_{ik}) + E_L a_i a_j a_m a_n d_{mn}, \tag{4}$$

with

$$d_{ii} = 0. \tag{5}$$

Here p still denotes the hydrostatic pressure, but η_L and η_T are the viscosities for shear flow respectively along and transverse to the fibres, and E_L is related to the extensional viscosity in the fibre-direction.

Equation (4) may be interpreted as the constitutive equation for a transversely isotropic fluid, with the fibre-direction $\mathbf{a}$ being the local axis of rotational symmetry. It should be noted that although the fibres may all be parallel and straight in the initial configuration, in general there is no constraint on their remaining so in the subsequent flow.

For the "ideal fibre-reinforced material" model, we also have local inextensibility in the $\mathbf{a}$-direction. Assuming convection of the fibres with the deformation, this implies that

$$a_i a_j d_{ij} = 0, \quad a_i a_i = 1 \tag{6}$$

and that the material time derivative of $\mathbf{a}$ is given by Spencer [1] as

$$\dot{a}_i = (\delta_{ij} - a_i a_j) a_k \partial v_j / \partial x_k = a_k \partial v_i / \partial x_k. \tag{7}$$

Furthermore, equation (4) is replaced by

$$\sigma_{ij} = -p\delta_{ij} + Ta_i a_j + 2\eta_T d_{ij} + 2(\eta_L - \eta_T)(a_i a_k d_{kj} + a_j a_k d_{ki}), \tag{8}$$

where T is an arbitrary stress reaction to the further internal constraint of fibre-inextensibility. Both T and p are arbitrary functions of $\mathbf{x}$ and t, being independent of the rate-of-strain field. They are, in effect, determined from the equations of motion, which in the absence of body forces take the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \varrho \dot{v}_i. \tag{9}$$

Equations (5)–(9) constitute the complete set of governing equations for an ideal fibre-reinforced linear viscous fluid.

For some problems the fibre-direction remains constant throughout the flow. In these cases, with $\mathbf{a} = (0, 0, 1)$, equations (5)–(8) simplify to give

$$d_{11} = -d_{22}, \quad d_{33} = 0 \tag{10}$$

and

$$\begin{aligned} \sigma_{11} &= -p + 2\eta_T d_{11}, \quad \sigma_{22} = -p - 2\eta_T d_{11}, \quad \sigma_{33} = -p + T, \\ \sigma_{12} &= 2\eta_T d_{12}, \quad \sigma_{13} = 2\eta_L d_{13}, \quad \sigma_{23} = 2\eta_L d_{23}. \end{aligned} \tag{11}$$

If furthermore no shear occurs parallel to the x_3-axis, then the flow is planar, with

$$\begin{aligned} v_1 &= v_1(x_1, x_2, t), \quad v_2 = v_2(x_1, x_2, t), \quad v_3 = 0, \\ \sigma_{11} &= -p + 2\eta_T \partial v_1/\partial x_1, \quad \sigma_{22} = -p - 2\eta_T \partial v_1/\partial x_1, \\ \sigma_{33} &= -p + T, \quad \sigma_{12} = \eta_T(\partial v_1/\partial x_2 + \partial v_2/\partial x_1), \quad \sigma_{13} = \sigma_{23} = 0, \end{aligned} \tag{12}$$

yielding the same equations as those for plane Newtonian flow [6].

3. Squeezing flow

We now consider the particular problem of squeezing an initially rectangular block of fluid between two rigid platens which are slowly brought together under normal loading. The fibres are initially straight and parallel, and lie in planes parallel to the platens.

The problem was first posed by Cogswell [3] who presented a simplified analysis to give an "apparent viscosity" η_A at the beginning of the flow, in terms of the geometry of the block, the load applied and the initial rate of closing-up of the platens. Further work, including experimental data, has been reported by Barnes [4], who modified Cogswell's analysis to take

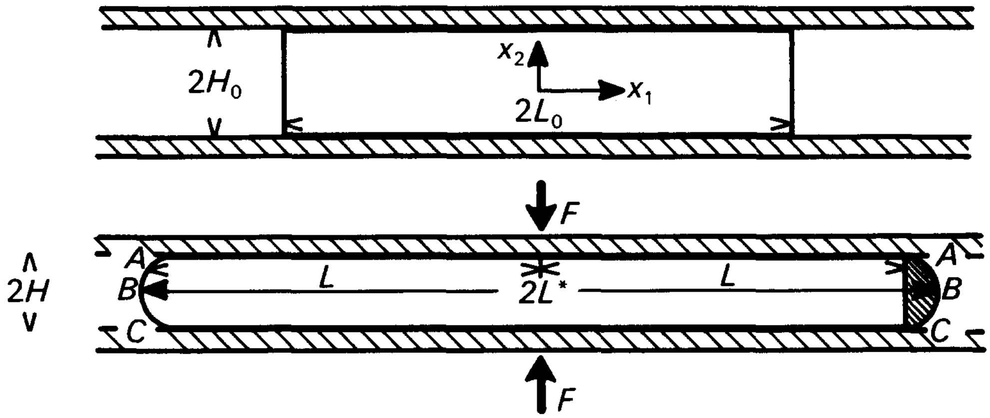

Fig. 1

account of continued flow, and by Jones and Wheeler [5]. The following analysis is more rigorous than that of [3] and [4], and allows more consideration of the effect of different slip conditions in the contact regions between fluid and platens. We believe that it also has the merit of the being derived from a self-consistent theory, namely that described above.

The axes are chosen such that the x_3-axis is parallel to the initial fibre-direction and the x_2-axis is perpendicular to the platens; the initial cross-section of the fluid (ref. Fig. 1) is denoted by

$$-L_0 \leqslant x_1 \leqslant L_0, \quad -H_0 \leqslant x_2 \leqslant H_0.$$

During the flow the thickness $2H$ of the fluid gradually decreases, and the boundary conditions at the rigid platens require that the normal velocity there is dependent only on the time, and independent of both x_1 and x_3:

$$v_2 = \pm \dot{H}(t) \quad \text{at} \quad x_2 = \pm H(t); \tag{13}$$

the superposed dot denotes differentiation with respect to time. Furthermore, continued contact with the platens implies that the two surface layers of fibres will remain straight and parallel through the flow. Finally we note that in practice the aspect ratio H_0/L_0 is small, and we assume this.

All these considerations suggest that a suitable choice of flow field is given by

$$v_1 = v_1(x_1, x_2, x_3, t), \quad v_2(x_2, t), \quad v_3 = v_3(x_1, x_2, x_3, t)$$

and

$$\mathbf{a} = (0, 0, 1).$$

The symmetry of the problem requires that v_1, v_2 and v_3 are odd functions of x_1, x_2 and x_3 respectively, with

$$v_1 = 0 \text{ on } x_1 = 0, \quad v_2 = 0 \text{ on } x_2 = 0, \quad v_3 = 0 \text{ on } x_3 = 0. \tag{14}$$

Hence inextensibility in the fibre-direction implies that

$$v_3 = 0$$

throughout the fluid. Then incompressibility gives

$$\frac{\partial v_1}{\partial x_1} = -\frac{\partial v_2}{\partial x_2} = -v_2' \quad \text{(say)}, \tag{15}$$

where the superposed dash indicates differentiation with respect to the single space coordinate x_2. Integration with respect to x_1, together with (14), yields

$$v_1 = -x_1 v_2'(x_2, t). \tag{16}$$

The flow is planar, and equations (12) take the form

$$\sigma_{11} = -p - 2\eta v_2', \quad \sigma_{22} = -p + 2\eta v_2', \quad \sigma_{33} = -p + T,$$

$$\sigma_{12} = -\eta x_1 v_2'', \quad \sigma_{13} = \sigma_{23} = 0. \tag{17}$$

Here, for convenience, η is used to denote η_T, the only viscosity in the analysis.

If we now make the usual assumption of creeping flows that the inertial terms may be neglected in the equations of motion (9), substitution of (17) into (9) and straightforward integration result in

$$p = p_0 + \eta(v_2' - Ax_1^2), \tag{18}$$

where

$$v_2 = \tfrac{1}{3}Ax_2^3 + Bx_2. \tag{19}$$

Here A, B and p_0 are functions of t only, and we have ensured that v_2 is an odd function of x_2. From (16) and (19) we obtain

$$v_1 = -(Ax_2^2 + B)x_1 \tag{20}$$

and

$$\sigma_{11} = -p_0 + \eta\{Ax_1^2 - 3(Ax_2^2 + B)\}, \quad \sigma_{12} = -2\eta A x_1 x_2,$$

$$\sigma_{22} = -p_0 + \eta\{Ax_1^2 + Ax_2^2 + B\}. \tag{21}$$

Zero traction on the two edges perpendicular to the fibre-direction gives

$$T = p.$$

The three arbitrary functions A, B and p_0 are determined by requiring that
(*i*) the edges $x_1 = \pm L$ are traction free,
(*ii*) the total compressive load F is specified on each platen $x_2 = \pm H$, and
(*iii*) the conditions in the contact region (e.g., no-slip, or negligible friction, etc.) are satisfied.

Equations (13) and (19) then provide a rate equation to relate F with $H(t)$:

$$\dot{H} = (\tfrac{1}{3}AH^2 + B)H. \tag{22}$$

The edge surfaces will vary with time since incompressibility together with inextensibility in the x_3-direction imply that the cross-sectional area is conserved (with value $4L_0H_0$) during the flow. In general they will also become curved, and the pointwise condition of zero traction will not be satisfied by the present simple form of flow field. However, we can satisfy global equilibrium of the edge regions of the fluid (such as the shaded area ABC in Fig. 1) provided

$$\int_{-H}^{H} \sigma_{11}(\pm L, x_2, t)\, \mathrm{d}x_2 = 0;$$

here $2L(t)$ denotes the total length of the contact region with each platen. Substituting from (21) then leads to

$$p_0 = -\eta(AH^2 + 3B - AL^2). \tag{23}$$

We note that the pointwise mismatch in the edge boundary tractions will lead to edge effects which should be small, especially for sufficiently small aspect ratios.

The compressive load, per unit length in the fibre-direction, is given by

$$F = -\int_{-L}^{L} \sigma_{22}(x_1, H, t)\, \mathrm{d}x_1 = 4\eta L(\tfrac{1}{3}AL^2 - AH^2 - 2B). \tag{24}$$

An alternative expression can be obtained from

$$F^* = -\int_{-L^*}^{L^*} \sigma_{22}(x_1, 0, t)\, \mathrm{d}x_1 = 2\eta L^*(AL^2 - AH^2 - \tfrac{1}{3}AL^{*2} - 4B). \tag{25}$$

Here $2L^*(t)$ is the maximum total width of the specimen, being the total width of the midplane $x_2 = 0$. It differs from $2L(t)$, with the "average" width $2\bar{L} = 2L_0H_0/H$ being intermediate between the two values. The two values F and F^* are not equal, except when $L = L^*$ (as initially, for instance); they differ by the x_2-component of the resultant traction on the upper (or, equivalently, the lower) surfaces of the edge regions ABC, necessary to maintain the predicted flow field. Again this gives rise to an edge effect, since the exact problem has zero traction on such boundaries. In any case we cannot expect the flow field (19) and (20) to be a good approximation to the flow in such edge regions, since the rigid platens are not in contact there.

Initially $L^* = L$, and although in general the difference $L^* - L$ will not remain zero, we expect that in practice it will be no more than $\mathrm{O}(H)$ so that

$$L = \bar{L} + \mathrm{O}(H) = L_0H_0/H + \mathrm{O}(H). \tag{26}$$

4. Zero-friction contact

The remaining condition to be satisfied is that specified at the platens. If the interfaces between fluid and platens are free from shear traction, such as by the effect of introducing a lubricating agent, then the contact condition is $\sigma_{12} = 0$ at $x_2 = \pm H$. From (21) and (24) this immediately implies that

$$A = 0, \quad B = -F/(8\eta L),$$

so that

$$v_1 = -Bx_1, \quad v_2 = Bx_2, \quad \sigma_{11} = \sigma_{12} = 0, \quad \sigma_{22} = -F/2L.$$

In fact, these give the expected simple solution of pure shear, in which the cross-sectional shape remains rectangular with

$$L = L_0 H_0 / H.$$

So now F and H are related through

$$F = -8\eta L_0 H_0 \dot{H}/H^2$$

so that, for a constant load F_0,

$$\frac{1}{H} - \frac{1}{H_0} = \frac{F_0 t}{8\eta L_0 H_0}. \tag{27}$$

5. No-slip contact

If we assume that no slip occurs between fluid and platen in the contact region $-L \leqslant x_1 \leqslant L$ at $x_2 = \pm H$, then v_1 is zero there. Hence, from (20) and (24),

$$A = \frac{3F}{4\eta L(L^2 + 3H^2)}, \quad B = -AH^2. \tag{28}$$

(i) $L(t) = L_0$; *constant contact region*

The velocity field $v_2 = v_2(x_2, t)$ implies that the flow cannot increase the contact region. Hence, from (22) the relation between F and H in the early stages of the developing flow is determined from

$$\dot{H} = -\frac{FH^3}{2\eta L_0(L_0^2 + 3H^2)}, \tag{29}$$

and hence

$$\frac{1}{H^2} - \frac{1}{H_0^2} - \frac{6}{L_0^2} \ln\left(\frac{H}{H_0}\right) = \frac{1}{\eta L_0^3} \int_0^t F(t')\, dt'. \tag{30}$$

For $H_0/L_0 \ll 1$, the log term is negligible, and for constant load F_0 equation (30) yields the simple relation

$$\frac{1}{H^2} - \frac{1}{H_0^2} \simeq \frac{F_0 t}{\eta L_0^3}. \tag{31}$$

We note also that, for $H_0/L_0 \ll 1$, the expression (29) yields the relation deduced by Cogswell [3] for the apparent viscosity at the beginning of the flow:

$$\eta_0 = -\frac{FH_0^3}{2L_0\dot{H}_0}, \quad \dot{H}_0 = \dot{H}(0).$$

(ii) $L(t) > L_0$; varying contact region

The tractions that need to be applied on the edge surfaces ABC in order to maintain the velocity field (19) indicate that in the actual situation of zero traction the developing edge region will spread onto the platens, thus increasing the contact region. Then, from (26),

$$L \simeq L_0 H_0/H$$

and (29) is replaced by

$$\dot{H} = -\frac{FH^3}{2\eta L(L^2 + 3H^2)} = -\frac{FH^6}{2\eta L_0 H_0(L_0^2 H_0^2 + 3H^4)}, \tag{32}$$

Now H and F are related through

$$\tfrac{1}{5} L_0^2 H_0^2 \left(\frac{1}{H^5} - \frac{1}{H_0^5}\right) + 3\left(\frac{1}{H} - \frac{1}{H_0}\right) = \frac{1}{2\eta L_0 H_0} \int_0^t F(t')\, dt'. \tag{33}$$

For $H_0/L_0 \ll 1$, the second term is negligible, and for constant load F_0, (33) simplifies to give

$$\frac{1}{H^5} - \frac{1}{H_0^5} \simeq \frac{5F_0 t}{2\eta L_0^3 H_0^3}. \tag{34}$$

Finally, we note that for $H_0/L_0 \ll 1$, expression (32) gives the relation given by Barnes [4] for the apparent viscosity of the developed flow.

6. Conclusions

This paper is an attempt to draw attention to the potential of the concept of highly anisotropic fluids for describing flows of fibre-reinforced composites at elevated temperatures. Whilst the solution presented for slow squeezing flow is based on the "ideal" model, which assumes fibre-inextensibility, a more general asymptotic solution for slightly-extensional, real composites can be derived for which the present solution would be the zeroth order approximation. Obviously such an approach is not restricted to this particular application; other problems can be treated in a similar manner within the same coherent mathematical framework.

Finally, in a practical context, we note the widely-different relations (27), (31) and (34) between the constant applied load F_0 and the resulting flow as indicated by the thickness variable $H(t)$. This disparity appears to agree with the observation that a change in the adhesion conditions at the contact surfaces can radically affect the flow.

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